Lecture 8

The Lévy-Itô decomposition

Theorem 1. *Given any $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and a Lévy measure ν, there exists a probability space on which three independent Lévy processes exist, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ where:*

- $X^{(1)}$ *is a Brownian motion with drift:*

$$X_t^{(1)} = \gamma t + \sigma W_t, \quad t \geq 0,$$

- $X^{(2)}$ *is a square integrable martingale with characteristic exponent*

$$\psi^{(2)}(u) = \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbf{1}_{|x|\leq 1})\nu(dx).$$

- $X^{(3)}$ *is a compound Poisson process:*

$$X_t^{(3)} = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda := \nu(\mathbb{R} \setminus (-1, 1))$ independent of the i.i.d. sequence $(\xi_i)_{i\geq 1}$ with distribution concentrated on the set $\{x : |x| \geq 1\}$ and given by $\nu(dx)/\lambda$ (unless $\lambda = 0$ in which case $X^{(3)}$ is the process which is identically zero).

By taking $X := X^{(1)} + X^{(2)} + X^{(3)}$ we see that there exists a probability space on which a Lévy process is defined with characteristic exponent

$$\psi(u) = -i\gamma u + \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbf{1}_{|x|\leq 1})\nu(dx).$$

In other words, the Lévy-Itô decomposition tells us that X is a Lévy process with Lévy triplet (γ, σ^2, ν) if and only if it can be written as the sum of three independent Lévy processes:

$$X_t = \gamma t + \sigma W_t + \lim_{\eta \to 0} \left(\sum_{s\leq t} \Delta X_s \mathbf{1}_{\eta < |\Delta X_s| \leq 1} - t \int_{\eta < |x| \leq 1} x\nu(dx) \right) + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| > 1},$$

where

- $W = (W_t)_{t\geq 0}$ is a standard Brownian motion;

- $(\sum_{s\leq t}\Delta X_s \mathbf{1}_{\eta<|\Delta X_s|\leq 1} - t\int_{\eta<|x|\leq 1} x\nu(dx))_{t\geq 0}$ converges in L_2, as η goes to zero, to a martingale denoted by $M=(M_t)_{t\geq 0}$ with characteristic function given by

$$\mathbb{E}[e^{iuM_t}] = \exp\bigg(-t\int_{|x|\leq 1}(1-e^{iux}+iux)\nu(dx)\bigg).$$

- $(\sum_{s\leq t}\Delta X_s \mathbf{1}_{|\Delta X_s|>1})_{t\geq 0}$ is a Lévy process with finite Lévy measure, i.e. it is a compound Poisson process with intensity $\lambda := \nu(\{x : |x|>1\})$ and jump distribution concentrated on the set $\{x : |x|>1\}$ and given by $\nu(dx)/\lambda$. In particular, its characteristic function is given by

$$\exp\bigg(-t\int_{|x|>1}(1-e^{iux})\nu(dx)\bigg).$$

- The processes $(\gamma t+\sigma W_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$ and $(\sum_{s\leq t}\Delta X_s \mathbf{1}_{|\Delta X_s|>1})_{t\geq 0}$ are three independent Lévy processes.

Nota Bene: In the general form of the Lévy-Itô decomposition one separates the big jumps $(\sum_{s\leq t}\Delta X_s \mathbf{1}_{|\Delta X_s|>1})_{t\geq 0}$ from the small jumps M since the infinite sum

$$\sum_{s\leq t}\Delta X_s \mathbf{1}_{|\Delta X_s|\neq 0}, \quad t\geq 0$$

is almost surely not defined for Lévy measures ν such that $\int_{-1}^{1}|x|\nu(dx)=\infty$. It can be shown that $\Big|\sum_{s\leq t}\Delta X_s\Big| < \infty$ a.s. whenever $\int_{-1}^{1}|x|\nu(dx)<\infty$. In particular, a purely discontinuous Lévy process X (also said *pure jumps Lévy process*) with a Lévy measure ν such that $\int_{-1}^{1}|x|\nu(dx)<\infty$ can be written as the sum of all its jumps, i.e.

$$X_t = \sum_{s\leq t}\Delta X_s \mathbf{1}_{|\Delta X_s|\neq 0}, \quad t\geq 0. \tag{1}$$

Observe that the Lévy triplet of X as in (1) is given by $(\int_{-1}^{1} x\nu(dx), 0, \nu)$, that is its characteristic function is given by

$$\exp\bigg(-t\int_{\mathbb{R}}(1-e^{iux})\nu(dx)\bigg).$$

Lévy triplet

As we have already seen, the law of a Lévy process X is uniquely determined by its Lévy triplet (γ,σ^2,ν). We will use the following terminology:

- γ: drift,
- σ^2: diffusion coefficient,
- ν: Lévy measure.

Keeping the Lévy-Itô decomposition in mind, it is clear that the drift and the diffusion coefficient characterize the continuous part of X whereas ν contains all the information about the jumps, i.e. it dictates the behavior of the discontinuous part of X. A way to better interpret the role played by the Lévy measure is offered by the following equality:

$$\nu(B) = \frac{1}{t}\mathbb{E}\Big[\sum_{0<s\le t} \mathbf{1}_B(\Delta X_s)\Big], \quad \forall B \in \mathscr{B}(\mathbb{R}),$$

that is the Lévy measure of any Borel set B represents the average number of jumps, per unit of time, the magnitudes of which fall in B.

Examples

- Brownian motion with drift: $X_t = \gamma t + \sigma W_t, \quad t \ge 0$.
  The Lévy triplet is given by $(\gamma, \sigma^2, 0)$.
- Poisson process: let N be a Poisson process of intensity λ, then its Lévy triplet is given by $(\lambda, 0, \lambda\delta_1)$.
- Compound Poisson process: $X_t = \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process of intensity λ independent of the sequence $(Y_i)_{i\ge 1}$ with distribution F. Then, the Lévy triplet of X is given by $(\lambda\int_{-1}^{1} xF(dx), 0, \lambda F)$.

Relationship between the Lévy measure of X and the law of X

Let X be a compound Poisson process with intensity λ and jump size distribution F. Denote by N_t the number of jumps of X on $[0,t]$. Then, for any Borel set A,

$$\begin{aligned}\mathbb{P}(X_t \in A) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t \in A | N_t = n)\mathbb{P}(N_t = n)\\ &= e^{-\lambda t}\delta_0 + \sum_{n=1}^{\infty} F^{*n}(A)\frac{e^{-\lambda t}(\lambda t)^n}{n!},\end{aligned}$$

where F^{*n} denotes the n-th convolution power of F and δ_0 stands for the Dirac measure at point 0. Let us denote by ν the Lévy measure of X, that is

$$\nu(A) = \lambda F(A) = \lambda\mathbb{P}(Y_1 \in A), \quad \forall A \in \mathscr{B}(\mathbb{R}).$$

In particular, for any Borel set A that does not contain 0, we have

$$\lim_{t\downarrow 0} \frac{\mathbb{P}(X_t \in A)}{t} = \lim_{t\downarrow 0}\bigg(\lambda\mathbb{P}(Y_1 \in A)e^{-\lambda t} + \lambda\sum_{n=2}^{\infty}\mathbb{P}(Y_1 + \cdots + Y_n \in A)\frac{e^{-\lambda t}(\lambda t)^{n-1}}{n!}\bigg) = \nu(A)$$

since

$$0 \le \lambda\sum_{n=2}^{\infty}\mathbb{P}(Y_1 + \cdots + Y_n \in A)\frac{e^{-\lambda t}(\lambda t)^{n-1}}{n!} \le \frac{e^{-\lambda t}}{t}\sum_{n=2}^{\infty}\frac{(\lambda t)^n}{n!} = \frac{e^{-\lambda t}}{t}(e^{\lambda t} - 1 - \lambda t) \xrightarrow[t\to 0]{} 0.$$

More generally, the following result can be proven.

Lemma 1. *Let X be a Lévy process with Lévy measure ν. If g is a function such that $\int_{|x|\geq 1} g(x)\nu(dx) < \infty$, $\lim_{x\to 0} \frac{g(x)}{x^2} = 0$ and $\frac{g(x)}{(|x|^2\wedge 1)}$ is bounded for all x in $\mathbb{R}$, then*

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}[g(X_t)] = \int_{\mathbb{R}} g(x)\nu(dx).$$

In particular, Lemma 1 applied to $g = \mathbf{1}_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}$ implies that

$$\nu(\mathbb{R} \setminus (-\varepsilon, \varepsilon)) = \lim_{t\to 0} \frac{1}{t}\mathbb{P}(|X_t| > \varepsilon), \quad \forall\ 0 \leq \varepsilon \leq 1.$$

Another important property of the Lévy measure is that it determines the tail behavior of the distribution of a Lévy process and its moments. Indeed, if X is a Lévy process with characteristic exponent ψ and $\mathbb{E}[|X_t|] < \infty$, then by differentiation one derives $\mathbb{E}[X_t] = it\psi'(0)$. Using the Lévy-Khintchine formula one can thus easily write the expectation of any Lévy process in terms of its drift and Lévy measure. In the same way, if $\mathbb{E}[|X_t|^2] < \infty$, then $\text{Var}(X_t) = t\psi''(0)$ and we can thus express the variance of X easily in terms of its diffusion coefficient and the Lévy measure. More precisely, the following result holds:

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (γ, σ^2, ν). The n-th absolute moment of X_t, $\mathbb{E}[|X_t|^n]$ is finite for some t or, equivalently, for every $t > 0$ if and only if $\int_{|x|\geq 1} |x|^n\nu(dx) < \infty$. In this case the moments of X_t can be computed from its characteristic function by differentiation. In particular*

$$\mathbb{E}[X_t] = t\left(\gamma + \int_{|x|\geq 1} x\nu(dx)\right),$$

$$Var(X_t) = t\left(\sigma^2 + \int_{\mathbb{R}} x^2\nu(dx)\right).$$

References

[1] Bertoin, J., Lévy Processes, Cambridge University Press, 1996.

[2] Cont, R., and Tankov, P., Financial Modelling With Jumps Processes, Chapman & Hall/CRC, 2004.

[3] Kyprianou, A., Fluctuations of Lévy Processes with Applications, Springer, 2104.

[4] Sato, K., Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.