Course *Statistics for Stochastic Processes*
Winter 2017/18
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Exercises: sheet 5

Oral presentation (By 21.12.2017)

Exercise 1. *Let $X = (X_t)_{t\geq 0}$ be a process such that $X_0 = 0$ a.s. and with independent increments, i.e.*

$$\forall n,\ \forall t_1 < t_2 < \cdots < t_n, \quad X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent random variables.

Show that for any $t > s$ the random variable $(X_t - X_s)$ is independent of the σ-algebra $\sigma(X_u : u \leq s)$.

Exercise 2. *Let X_1 and X_2 be two independent Lévy processes with Lévy triplets $(\gamma_1, \sigma_1^2, \nu_1)$ and $(\gamma_2, \sigma_2^2, \nu_2)$. Prove that $X_1 + X_2$ is a Lévy process and specify its Lévy triplet.*

[Two stochastic processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are independent if, for all $m, n \in \mathbb{N}$, all $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and all $0 \leq s_1 < s_2 < \cdots < s_n < \infty$ the σ-algebras $\sigma(X_{t_1}, \ldots, X_{t_n})$ and $\sigma(Y_{s_1}, \ldots, Y_{s_m})$ are independent.]

Exercise 3. *Let X be a Lévy process with Lévy triplet (γ, σ^2, ν) with $\int_{|x|\geq 1} |x|^3 \nu(dx) < \infty$. Compute $\mathbb{E}[X_t^3]$.*

Written production (By 21.12.2017)

Exercise 4. *Let X be a Lévy process with characteristic exponent ψ. Show that ψ determines the law of X.*

Exercise 5. *Let X be a compound Poisson process:*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process independent of the i.i.d. sequence $(Y_i)_{i\geq 1}$.

- *Prove that X is a Lévy process.*
- *Prove that the characteristic exponent of X is given by*

$$\psi(u) = \lambda(1 - \mathbb{E}[e^{iuY_1}]).$$