Course *Statistics for Stochastic Processes*
Winter 2017/18
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Exercises: sheet 4

Oral presentation (By 07.12.2017)

Exercise 1. *Let* $(X_t)_{t\in\mathbb{Z}}$ *be a causal* $AR(p)$ *of the form*

$$X_{t+1} = \phi_1 X_t + \cdots + \phi_p X_{t-p+1} + Z_{t+1}, \quad Z \sim IID(0, \sigma^2)$$

and let C_p *be the covariance matrix of* $(X_1, \ldots, X_p)$*. Prove that* C_p *is non-singular.*

Exercise 2. *Let* X *be a* $MA(1)$*:*

$$X_t = Z_t - \alpha Z_{t-1}, \quad Z \sim IID(0, \sigma^2)$$

with $|\alpha| < 1$*. Prove that the best predictor of* X_{t+1} *given* $(X_{t-k})_{k\geq 0}$*, denoted by* $\widehat{X}_{t+1}$*, is*

$$\widehat{X}_{t+1} = -\sum_{j=1}^{\infty} \alpha^j X_{t+1-j}.$$

Hint: Assume the fact that for $|\alpha| < 1$ *the following representation holds:* $Z_t = \sum_{j=0}^{\infty} \alpha^j X_{t-j}$*.*

Exercise 3. *Let* $Y_t = \theta + X_t$ *where* $X = (X_t)_{t\in\mathbb{Z}}$ *is an* $AR(1)$ *defined as*

$$X_t - \phi X_{t-1} = Z_t, \quad |\phi| < 1, \quad Z \sim IID(0, \sigma^2).$$

Let $\widehat{\theta}_n$ *be the sample mean of* $Y_0, \ldots, Y_{n-1}$*:* $\widehat{\theta}_n = \frac{1}{n}\sum_{i=0}^{n-1} Y_i$*.*

1. *Compute* $\lim_{n\to\infty} n\,Var(\widehat{\theta}_n)$*. Admit the asymptotic normality of* $\sqrt{n}(\widehat{\theta}_n - \theta)$ *and give the expression for the asymptotic variance* γ*, i.e.* γ *such that*

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \gamma).$$

2. *Suppose that* $\phi = 0.6$*,* $\sigma^2 = 2$ *and* $\widehat{\theta}_{100} = 0.271$*. Construct an asymptotic confidence interval for* θ *at level 95%. Could you accept the hypothesis that* $\theta = 0$*?*

Written production (By 07.12.2017)

Exercise 4. *Let $X^2, Z \in L_2$ be two independent random variables with $\mathbb{E}[Z] = 0$ and $Var(X) \neq 0$. Set $Y = X^2 + Z$.*

1. *Compute $\mathbb{E}[Y|X]$.*
2. *Compute the best linear predictor $Y^* = a + bX$, for some $a, b \in \mathbb{R}$, of Y given X.*

Suppose that X and Z are centered Gaussian distributions with unit variance.

3. *Prove that the square predictor error of Y^* is strictly bigger than the square predictor error of $\mathbb{E}[Y|X]$.*

Exercise 5. *Let X be an $AR(2)$ defined as*

$$X_t = 0.4X_{t-1} - 0.2X_{t-2} + Z_t, \quad Z \sim WN(0, 12.8).$$

1. *Verify that X is weakly stationary.*
2. *Write the Yule-Walker equations.*

Exercise 6. *Let X be an $AR(p)$ defined by*

$$X_{t+1} = \phi_1 X_t + \cdots + \phi_p X_{t-p+1} + Z_{t+1}, \quad Z \sim WN(0, \sigma^2)$$

1. *Compute the best predictor $\widehat{X}_{t+1}$ of X_{t+1} given $X_t, \ldots, X_{t+1-p}$.*
2. *Compute the best predictor $\widehat{X}_{t+h}$ of X_{t+h} given $X_t, \ldots, X_{t+1-p}$.*
3. *Compute the square predictor error.*