Course *Statistics for Stochastic Processes*
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## Exercises: sheet 3

### Oral presentation (By 23.11.2017)

**Exercise 1.** *Let* $(\alpha_k)_{k\in\mathbb{Z}}$ *and* $(\beta_k)_{k\in\mathbb{Z}}$ *be sequences of real numbers such that* $\sum_{k\in\mathbb{Z}}|\alpha_k|<\infty$ *and* $\sum_{k\in\mathbb{Z}}|\beta_k|<\infty$*. Prove that if* $X\in\mathcal{S}(\Omega,\mathcal{A},\mathbb{P})$ *then*

$$F_\alpha[F_\beta[X]] = F_{\alpha\beta}[X], \quad \textit{where} \;\; (\alpha\beta)_k := \sum_{j\in\mathbb{Z}}\alpha_j\beta_{k-j}.$$

**Exercise 2.** *Let* $X$ *be a weakly stationary ARMA*$(p,q)$*-process on* $\mathbb{Z}$*:*

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \tag{1}$$

*where* $Z\sim WN(0,\sigma^2)$ *and* $\phi_1,\dots,\phi_p,\theta_1,\dots,\theta_q$ *are real numbers. Introduce the polynomials* $\phi(z)=1-\phi_1 z-\cdots-\phi_p z^p$ *and* $\theta(z)=1+\theta_1 z+\cdots+\theta_q z^q$ *and suppose that* $\phi$ *and* $\theta$ *have no common roots on* $\{z\in\mathbb{C}:|z|\le 1\}$*. Prove that* $X$ *is causal if and only if* $\phi(z)\neq 0$ *for* $z\in\mathbb{C}$ *with* $|z|\le 1$*. In that case* $X_t=\sum_{j=0}^\infty \psi_j Z_{t-j}$ *holds where* $\psi(z):=\sum_{j=0}^\infty\psi_j z^j=\frac{\theta(z)}{\phi(z)}$ *for* $|z|\le 1$*.*

**Exercise 3** (GARCH processes)**.** *A time series* $(X_t)_{t\in\mathbb{Z}}$ *is called a GARCH*$(1,1)$ *process if, for given nonnegative constants* $\alpha$*,* $\theta$ *and* $\phi$*, and a given i.i.d. sequence* $(Z_t)_{t\in\mathbb{Z}}$ *with mean zero and unit variance, it satisfies the system of equations*

$$\begin{aligned} X_t &= \sigma_t Z_t,\\ \sigma_t^2 &= \alpha + \phi\sigma_{t-1}^2 + \theta X_{t-1}^2, \end{aligned}$$

*with*

*(i)* $\sigma_t\in L_2$ *is a measurable function of* $X_{t-1},X_{t-2},\dots;$

*(ii)* $Z_t$ *is independent of* $X_{t-1},X_{t-2},\dots.$

*Show that* $X$ *is a martingale difference series for the filtration* $\mathscr{F}_t=\sigma(X_t,X_{t-1},\dots)$*.*

**Exercise 4.** *Let* $(X_t)_{t\in\mathbb{Z}}$ *be a martingale difference series relative to the filtration* $\mathscr{F}_t$ *that satisfies the following properties:*

1. *There exists a positive constant $v$ such that*

$$n^{-1}\sum_{t=1}^{n}\mathbb{E}[X_t^2|\mathscr{F}_{t-1}] \xrightarrow[n\to\infty]{\mathbb{P}} v.$$

2. *For all $\varepsilon > 0$*

$$n^{-1}\sum_{t=1}^{n}\mathbb{E}[X_t^2\mathbf{1}_{|X_t|>\varepsilon\sqrt{n}}|\mathscr{F}_{t-1}] \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

*Consider the events $A_t = \left\{\frac{1}{n}\sum_{j=1}^{t}\mathbb{E}[X_j^2|\mathscr{F}_{j-1}] \leq 2v\right\}$ and set $X_{n,t} = \frac{1}{\sqrt{n}}X_t\mathbf{1}_{A_t}$. Prove that:*

- *The random variables $(X_{n,t})_{t\in\mathbb{N}}$ are martingale differences relative to the filtration $\mathscr{F}_t$.*
- *The random variables $X_{n,t}$ satisfy the following properties:*

$$\sum_{t=1}^{n}\mathbb{E}[X_{n,t}^2|\mathscr{F}_{t-1}] \leq 2v;$$

$$\sum_{t=1}^{n}\mathbb{E}[X_{n,t}^2|\mathscr{F}_{t-1}] \xrightarrow[n\to\infty]{\mathbb{P}} v;$$

$$\sum_{t=1}^{n}\mathbb{E}[X_{n,t}^2\mathbf{1}_{|X_{n,t}|>\varepsilon}|\mathscr{F}_{t-1}] \xrightarrow[n\to\infty]{\mathbb{P}} 0, \quad \forall\varepsilon > 0.$$

## Written production (By 23.11.2017)

**Exercise 5.** *Let $Z$ be a white noise. For each of the following equations, does it exist a weakly stationary solution? If yes, is it causal?*

1. $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$.
2. $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$.
3. $X_t + 0.6X_{t-2} = Z_t + 1.2Z_{t-1}$.
4. $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.

**Exercise 6.** *Let $(\psi_k)_{k\in\mathbb{Z}}$ be a sequence such that $\sum_{k\in\mathbb{Z}}|\psi_k| < \infty$ and let $(X_t)_{t\in\mathbb{Z}}$ be a weakly stationary process with mean $\mu_X$ and autocovariance function $c_X$. Prove that the process*

$$Y_t = \sum_{k\in\mathbb{Z}}\psi_k X_{t-k}$$

*is weakly stationary with mean $\mu_Y = \mu_X\sum_{k\in\mathbb{Z}}\psi_k$ and autocovariance function*

$$c_Y(h) = \sum_{j,k\in\mathbb{Z}}\psi_j\psi_k c_X(h+k-j).$$