Course *Statistics for Stochastic Processes*
Winter 2017/18
Prof. Markus Reiß
Dr Ester Mariucci
Humboldt-Universität zu Berlin

Exercises: sheet 1

Oral presentation (By 26.10.2017)

Exercise 1. *Prove that:*

- *If $X = (X_t)_{t \in T}$ are i.i.d., then X is strictly stationary.*
- *If X is in L_2 (i.e. $\mathbb{E}[X_t^2] < \infty$) and it is strictly stationary, then X is weakly stationary.*
- *Weakly stationarity does not in general imply strictly stationarity.*
- *Show that for a Gaussian process both notion of stationarity are equivalent.*

Exercise 2.

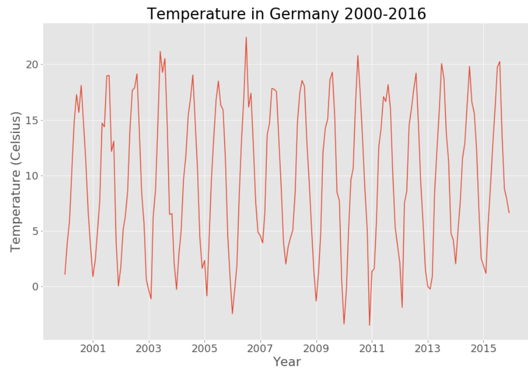
1. *Let $Y = (Y_t)_{t \in \mathbb{N}}$ be a weakly stationary process. Define*
$$X_t = \begin{cases} Y_t & t \text{ is even,} \\ Y_t + 1 & t \text{ is odd.} \end{cases}$$
Is X a weakly stationary process?
2. *Let $X = (X_t)_{t \in \mathbb{Z}}$ and $Y = (Y_t)_{t \in \mathbb{Z}}$ be two uncorrelated ($Cov(X_t, Y_s) = 0$, for all s, t) weakly stationary processes. Show that the process $Z_t = X_t + Y_t$ is weakly stationary. How does its autocovariance function can be derived from X and Y?*
3. *Define $X_t = A\cos(\theta t) + B\sin(\theta t)$ with A and B centered, independent and of variance σ^2 and θ a constant. Show that $X = (X_t)_{t \in \mathbb{Z}}$ is weakly stationary and compute the autocovariance function.*

Exercise 3. *Let $Z = (Z_t)_{t \in T}$ be a sequence of i.i.d. centered variables with finite variance σ^2 and θ a number. Determine if the following processes are weakly stationary or not and compute the mean and the autocovariance function in the affirmative case.*

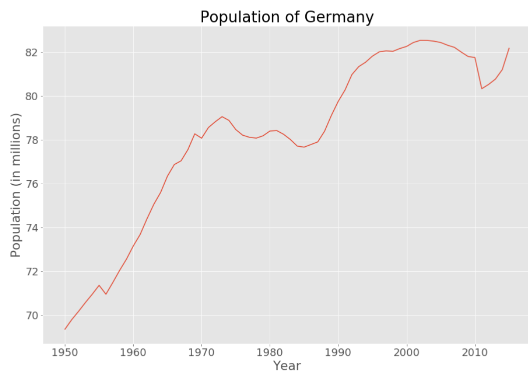
- $X_t = Z_t + \theta Z_{t-1}$, $t \in \mathbb{Z}$.
- $S_t = Z_1 + \cdots + Z_t$, $t \in \mathbb{N}$.
- $X_t = Z_t Z_{t-1}$, $t \in \mathbb{Z}$.

Written production (By 02.11.2017)

Exercise 4. *Is the following a representation of a stationary time series? Why or why not?*



And the following?



Exercise 5.

1. *Verify the bias-variance decomposition for an estimator* $\widehat{\theta}$ *of* $\theta \in \mathbb{R}$ *with* $\mathbb{E}[\widehat{\theta}] < \infty$*:*

$$\mathbb{E}[(\widehat{\theta} - \theta)^2] = \big(\mathbb{E}[\widehat{\theta}] - \theta\big)^2 + Var(\widehat{\theta}).$$

2. *Let* $Y_1, \ldots, Y_n$ *be i.i.d. Gaussian random variables with mean* μ *and variance* σ^2 *and denote by* $\widehat{\mu}_n$ *the empirical mean. Define*

$$\widehat{\sigma}_\alpha^2 := \frac{\alpha}{n-1} \sum_{i=1}^{n} (Y_i - \widehat{\mu}_n)^2, \quad \alpha > 0.$$

Show that $\widehat{\sigma}_\alpha^2$ *is unbiased if and only if* $\alpha = 1$ *and determine* $\alpha = \alpha_{opt} > 0$ *such that* $\mathbb{E}[(\widehat{\sigma}_\alpha^2 - \sigma^2)^2]$ *is minimal.*

*** Exercise 6.** *Let $p \in [1, \infty)$ and let $X = (X_n)_{n\in\mathbb{N}}$ be a strictly stationary process with $X_n \in L_p$ for any n. Then,*

$$\widehat{\mu}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$$

converges almost surely and in L_p.

***[Hint]** The following lemma might be useful:*

Lemma (Maximal lemma for stationary processes)**.** *Let $X = (X_n)_{n\in\mathbb{N}}$ be a strictly stationary process and let*

$$Y_n = X_0 + \cdots + X_n, \quad \Lambda = \cup_{n\in\mathbb{N}}\{Y_n > 0\}.$$

Then, $\int_\Lambda X_0 d\mathbb{P} \geq 0$.

To prove the convergence a.s.:

- *Show that $\liminf_n |\widehat{\mu}_n| < \infty$ almost surely.*
- *Observe that the set of points where $\widehat{\mu}_n$ is not pointwise convergent is the countable union of the events*

  $$S_{ab} = \left\{ \liminf_{n\to\infty} \widehat{\mu}_n < a, \ \limsup_{n\to\infty} \widehat{\mu}_n > b \right\}, \ a < b, \ a, b \in \mathbb{Q}.$$

  Let $a, b \in \mathbb{Q}$ with $a < b$, set $S = S_{ab}$ and define

  $$X'_n = (X_n - b)\mathbf{1}_S, \quad X''_n = (a - X_n)\mathbf{1}_S.$$

  Prove that $(X'_n)_{n\in\mathbb{N}}$ and $(X''_n)_{n\in\mathbb{N}}$ are strictly stationary.
- *Using the maximal lemma show that*

  $$\int_S X_0 d\mathbb{P} - b\mathbb{P}(S) = \int_S X'_0 d\mathbb{P} \geq 0,$$
  $$a\mathbb{P}(S) - \int_S X_0 d\mathbb{P} = \int_S X''_0 d\mathbb{P} \geq 0.$$
- *Deduce that $\mathbb{P}(S_{ab}) = 0$ for all $a < b$, $a, b \in \mathbb{Q}$ and therefore that $\widehat{\mu}_n$ converges almost surely.*

Convergence in L_p: Without loss of generality one can assume $\mathbb{E}[X_t] = 0$. Fix $\varepsilon > 0$ and let k be such that $\int_{|X_0|>k} |X_0|^p d\mathbb{P} \leq \left(\frac{\varepsilon}{4}\right)^p$. Write:

$$X_n = X'_n + X''_n, \quad \textit{where } X'_n = \max(-k, \min(k, X_n)).$$

Set $\widehat{\mu}'_n = \frac{1}{n}\sum_{i=1}^{n} X'_i$ and $\widehat{\mu}''_n = \frac{1}{n}\sum_{i=1}^{n} X''_i$.

- *Show that $\widehat{\mu}'_n$ converges in L_p to 0.*
- *Show that $\mathbb{E}[\widehat{\mu}''_n] \leq \left(\frac{\varepsilon}{4}\right)^p$.*
- *Using the completeness of L_p spaces conclude that $\widehat{\mu}_n$ converges in L_p to 0.*