Lecture 7

Stochastic processes in continuous time

As we have already discussed, a stochastic process is a family $X = (X_t)_{t\in T}$ of random variables indexed by the time. A possible definition is the following:

Definition 1. *Let T be a time set and $(E, \mathcal{E})$ a measurable space. A* stochastic process *indexed by T, taking values in $(E, \mathcal{E})$, is a collection $X = (X_t)_{t\in T}$ of measurable maps X_t from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the* state space *of the process.*

The time parameter t may be either discrete or continuous: until now we have focused on the discrete case, from now on we will assume the time to be continuous. Roughly speaking, there are three main objects that intervene in definition of a stochastic process X: the time set T, the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and the state space $(E, \mathcal{E})$ and thus three possible ways of seeing a stochastic process in continuous time:

1. For a fixed $t \in T$, $X_t : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathscr{E})$ is a random variable.
2. For a fixed $\omega \in \Omega$, $X_\cdot(\omega) : T \to (E, \mathscr{E})$ is a random function.
3. Considering $X : T \times \Omega \to (E, \mathscr{E})$ as a function of both the time t and the randomness ω.

The first interpretation is clear and corresponds to Definition 1. To be able to define a stochastic process as a function-valued random variable we need to define measures on function spaces. Probably, the first instinct would be to say that for a fixed ω, $X_\cdot(\omega)$ is a random variable on the space of all functions $f : T \to E$. However, this space happens to be in general too large, with too many pathological functions that make it hard to construct a measure on it. If we are interested on real stochastic processes with continuous sample paths, i.e. such that the trajectory $X_\cdot(\omega) : t \to X_t(\omega)$ is a continuous function of the time, then we can consider the space of continuous functions $C(T, \mathbb{R})$ with the topology induced by the sup norm

$$\|f\|_\infty = \sup_{t\in T} |f(t)|$$

to construct a Borel σ-algebra on which measures can be defined (for instance, the Wiener measure). However, this space is not always rich enough for our scopes and we would like to have a space that allows for discontinuous functions. More precisely, we will consider the class of real càdlàg functions (càdlàg: "continu à droite, limite à gauche"), that is functions that are right continuous with left limits. Finally, a stochastic process can also be seen as a function $X : T \times \Omega \to (E, \mathcal{E})$. Adopting this point of view means that a notion of jointly measurability must be introduced. This leads to the concepts of optional and predictable processes that are beyond the scope of these lectures.

Let us also recall that under some assumption on the state space, for instance it is enough to ask that E is a Polish space (a complete, separable metric space) and $\mathcal{E}$ is its Borel σ-algebra, i.e. the σ-algebra generated by the open sets, the law of the process X is uniquely determined by its finite-dimensional distributions:

Definition 2. *Let $X = (X_t)_{t \in T}$ be a stochastic process. The distributions of the finite-dimensional vectors of the form $(X_{t_1}, \dots, X_{t_n})$ are called the* finite-dimensional distributions *of the process.*

Càdlàg functions

Definition 3. *A function $f : [0, T] \to \mathbb{R}$ is said to be* càdlàg *if it is right-continuous and it admits left limits: for each $t \in [0, T]$ the limits*

$$f(t_-) = \lim_{s \uparrow t} f(s), \qquad f(t_+) = \lim_{s \downarrow t} f(s)$$

exist and $f(t) = f(t_+)$.

Of course, any continuous function is càdlàg but càdlàg functions can have discontinuities. If t is a discontinuity point we denote by

$$\Delta f(t) := f(t) - f(t_-)$$

the "jump" of f at time t. Here some properties of càdlàg functions that will be useful in understanding càdlàg processes.

1. Càdlàg functions can have at most a countable number of discontinuities, i.e. the set $\{t \in [0, T] : f(t) \neq f(t_-)\}$ is finite or countable.

2. For any $\varepsilon > 0$, the number of discontinuities ("jumps") on $[0, T]$ larger than ε is finite.

3. It follows that, for any $\varepsilon > 0$, a càdlàg function on $[0, T]$ has a finite number of "large jumps" (larger than ε) and a possibly infinite, but countable number of small jumps.

4. It is possible to define a topology and a notion of convergence on the space of càdlàg functions. Equipped with this topology and the corresponding Borel σ-algebra, the space of cádlág functions is known as the Skorokhod space and denoted by $D([0, T], \mathbb{R})$.

Definition 4. *A random variable with values in $D([0, T], \mathbb{R})$ is called a càdlàg process.*

Examples: Brownian motion and Poisson process

Definition 5. *A real-valued process $W = (W_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a* Brownian motion *if the following hold:*

(i) The trajectories of W are $\mathbb{P}$-almost surely continuous.

(ii) $\mathbb{P}(W_0 = 0) = 1$.

(iii) For $0 \leq s \leq t$, $W_t - W_s$ is independent of $\sigma(W_u : u \leq s)$.

(iv) For $0 \leq s \leq t$, $W_t - W_s$ is equal in distribution to W_{t-s}.

(v) For each $t > 0$, W_t is equal in distribution to a centered normal random variable with variance t.

Definition 6. *A process valued on the non-negative integers $N = (N_t)_{t\geq 0}$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:*

(i) The trajectories of N are $\mathbb{P}$-almost surely càdlàg.

(ii) $\mathbb{P}(N_0 = 0) = 1$.

(iii) For $0 \leq s \leq t$, $N_t - N_s$ is independent of $\sigma(N_u : u \leq s)$.

(iv) For $0 \leq s \leq t$, $N_t - N_s$ is equal in distribution to N_{t-s}.

(v) For each $t > 0$, N_t is equal in distribution to a Poisson random variable with parameter λt.

An equivalent definition of the Poisson process is the following.

Definition 7. *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n := \sum_{i=1}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t := \sum_{n=1}^{\infty} \mathbf{1}_{t \geq T_n}$$

is called a Poisson process with intensity λ.

At first glance, these two processes may appear very different. For instance: Brownian motion has continuous paths whereas a Poisson process does not; a Poisson process has paths of bounded variation over finite time horizons whereas a Brownian motion does not; Poisson processes are non-decreasing processes whereas a Brownian motion does not have monotone paths.

Looking better, however, we notice that they also have a lot in common. Both processes are càdlàg, are a.s. 0 at time 0 and both have stationary and independent increments. We may use these common properties to define a general class of processes called Lévy processes.

Lévy processes

Definition 8. *A càdlàg process $X = (X_t)_{t \geq 0}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $X_0 = 0$ a.s. is called a* Lévy process *if it possesses the following properties:*

1. *Independent increments: for every $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.*

2. *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.*

3. *Stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.*

Remark 1. *Some authors when defining a Lévy process X do not ask X to be càdlàg. However, since it can been shown that every Lévy process (defined without the càdlàg property) has a unique modification that is càdlàg, the càdlàg property can be assumed without loss of generality.*

Remark 2. *The third property does not imply that the sample paths of X are continuous and it is satisfied, for instance, by a Poisson process. It means that for a given time t, the probability of seeing a jump at t is zero; in other words, discontinuities may occur but only at random times. Processes with jumps at fixed times are excluded from the definition of Lévy processes.*

Lévy processes and infinitely divisible distributions

A very important feature of Lévy processes is the following. For any $n = 1, 2, \ldots$

$$X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_t - X_{\frac{(n-1)t}{n}}\right). \tag{1}$$

Since X has independent and stationary increments, from (1) it follows that the law of X_t is the convolution of n times the law of $X_{\frac{t}{n}}$, i.e. X_t can be divided into n i.i.d. parts. A distribution having this property is said to be infinitely divisible.

Definition 9. *A probability distribution F is said to be* infinitely divisible *if for any positive integer n, there exist n i.i.d. random variables $Y_1, \ldots, Y_n$ such that $Y_1 + \cdots + Y_n$ has distribution F.*

The notion of infinitely divisible distribution was introduced by De Finetti (in 1929) and it is intimately linked with that of Lévy processes. In fact, not only any marginal at time t of a Lévy process is infinitely divisible but the converse is also true. More precisely, the following result holds:

Theorem 1. *Let $X = (X_t)_{t \geq 0}$ be a Lévy process. Then for every t, X_t has an infinitely divisible distribution. Conversely, if F is an infinitely divisible distribution then there exists a Lévy process X such that the distribution of X_1 is given by F.*

The class of infinitely divisible distributions (and hence of Lévy processes) is very large, the most common examples being: the Gaussian distribution, the Poisson distribution, the

Gamma distribution, α-stable distributions and also log-normal, Pareto and Student distributions. A beautiful result that allows to describe the characteristic function of any Lévy process in terms of only three quantities (called, as we will see, the Lévy triplet) is the Lévy-Khinchine formula, proved by Paul Lévy and A. Ya. Khintchine in the 1930s. Before stating such a result, let us notice that the characteristic function of any Lévy processes can be written in the following particular way.

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. There exists a continuous function $\psi : \mathbb{R} \to \mathbb{C}$, $\psi(0) = 0$, such that*

$$\mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}.$$

The function ψ is called the characteristic exponent *of X.*

Proof. Let us denote by $\varphi(t) = \varphi(t, u)$ the characteristic function of X at time t and at point u: $\varphi(t) := \mathbb{E}[e^{iuX_t}]$. For any $s, t \geq 0$ we can write

$$X_{t+s} = (X_{t+s} - X_s) + X_s.$$

Using the fact that $X_{t+s} - X_s$ is independent of X_s and it is equal in law to X_t, we deduce that

$$\varphi(t + s) = \varphi(t)\varphi(s) \qquad \forall s, t \geq 0.$$

Moreover, from the stochastic continuity of X (i.e. $X_{t_n} \xrightarrow[t_n \to t]{\mathbb{P}} X_t$) it follows that $\varphi(t_n) \xrightarrow[t_n \to t]{} \varphi(t)$ that is the function $t \to \varphi(t)$ is continuous with $\varphi(0) = \mathbb{E}[e^{iuX_0}] = 1$. Since the only continuous solution to

$$\begin{aligned} \varphi(0) &= 1 \\ \varphi(t + s) &= \varphi(t)\varphi(s) \quad \forall s, t > 0 \end{aligned}$$

are given by $\varphi(u, t) = e^{-t\psi(u)}$ for some continuous function $\psi : \mathbb{R} \to \mathbb{C}$ such that $\psi(0) = 0$, the result is proven. □

Proposition 2. *The characteristic exponent ψ determines the law of the Lévy process X.*

Proof. Left for exercise. □

Examples

1. Brownian motion with drift:

$$X_t = \gamma t + \sigma W_t, \quad \sigma > 0, \tag{2}$$

   where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. Then X is a Lévy process with characteristic function given by

$$\mathbb{E}\big[e^{iuX_t}\big] = e^{iu\gamma t}\mathbb{E}\big[e^{iu\sigma W_t}\big] = e^{iu\gamma t} e^{-t\frac{u^2\sigma^2}{2}} = e^{-t(-iu\gamma + \frac{u^2\sigma^2}{2})},$$

   that is a Lévy process with characteristic exponent $\psi(u) = -iu\gamma + \frac{\sigma^2 u^2}{2}$.

   It can be proven that X defined as in (2) is the only Lévy process with a.s continuous trajectories.

2. Poisson process: Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity λ. Using the fact that the characteristic function of a Poisson r.v. Z with mean λ is given by $e^{-\lambda(1-e^{iu})}$ we deduce that
$$\mathbb{E}[e^{iuN_t}] = e^{-\lambda t(1-e^{iu})},$$
i.e. the characteristic exponent of a Poisson process with intensity λ is given by $\psi(u) = \lambda(1-e^{iu})$.

3. Compound Poisson process:
$$X_t = \sum_{i=1}^{N_t} Y_i,$$
where $N = (N_t)_{t\geq 0}$ is a Poisson process independent of the i.i.d. sequence $(Y_i)_{i\geq 1}$. Then X is a Lévy process with characteristic exponent given by
$$\psi(u) = \lambda(1 - \mathbb{E}[e^{iuY_1}]).$$

Lévy-Khintchine formula

Finally, let us state the well known Lévy-Khintchine formula, a crucial result on the understanding of Lévy processes.

Theorem 2. *$X = (X_t)_{t\geq 0}$ is a Lévy process if and only if there exist $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and a Borel measure ν concentrated on $\mathbb{R}\setminus\{0\}$ satisfying*
$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$$
such that $\mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}$ with
$$\psi(u) := -iu\gamma + \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{iux} + iux\mathbf{1}_{|x|\leq 1})\nu(dx).$$

The proof of this theorem is rather complicated and unfortunately we will not have the time to detail it. Let us however take the time to comment on the result.

- Theorem 2 implies that the law of a Lévy process is uniquely determined by the triplet (γ, σ^2, ν) that is called the *Lévy triplet*. As the previous examples suggest, γ and σ^2 come from a Brownian motion with drift whereas the measure ν, called *Lévy measure*, describes the jump part of the process.

- What does it mean the condition $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\nu(dx) < \infty$? Analytically speaking it means that
$$\nu(\{|x| \geq 1\}) < \infty \quad \text{and} \quad \int_{-1}^{1} x^2 \nu(dx) < \infty. \tag{3}$$
So, in particular, it follows that the integral in the Lévy Khintchine formula is finite since the integrand is $O(1)$ for $|x| \geq 1$ and $O(x^2)$ for $|x| < 1$.

- Condition (3) does not imply that ν is a finite measure! We may have $\nu((-1,1)) = \infty$ but in this case (3) implies that
$$\nu(\{|x| \in (\varepsilon, 1)\}) < \infty \quad \text{but} \quad \nu((-\varepsilon, \varepsilon)) = \infty \quad \forall\ 0 < \varepsilon < 1.$$

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