

Lecture 3

Theorem 1. Let $(\psi_k)_{k \in \mathbb{Z}}$ be a sequence such that $\sum_{k \in \mathbb{Z}} |\psi_k| < \infty$ and let $(X_t)_{t \in \mathbb{Z}}$ be a weakly stationary process with mean μ_X and autocovariance function c_X . Then the process

$$Y_t = \sum_{k \in \mathbb{Z}} \psi_k X_{t-k}$$

is weakly stationary with mean $\mu_Y = \mu_X \sum_{k \in \mathbb{Z}} \psi_k$ and autocovariance function

$$c_Y(h) = \sum_{j, k \in \mathbb{Z}} \psi_j \psi_k c_X(h + k - j).$$

Proof. Left as an exercise. □

Backshift calculus

Let $\mathcal{S}(\Omega, \mathcal{A}, \mathbb{P})$ be the space of all weakly stationary process $(X_t)_{t \in \mathbb{Z}}$. For any sequence $(\alpha_k)_{k \in \mathbb{Z}}$ such that $\sum_{k \in \mathbb{Z}} |\alpha_k| < \infty$ we define the operator F_α that at every $X \in \mathcal{S}(\Omega, \mathcal{A}, \mathbb{P})$ associates Y as follows:

$$F_\alpha : X \rightarrow Y := (Y_t)_{t \in \mathbb{Z}} = \left(\sum_{k \in \mathbb{Z}} \alpha_k X_{t-k} \right)_{t \in \mathbb{Z}}.$$

Thanks to Theorem 1, $F_\alpha : \mathcal{S}(\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathcal{S}(\Omega, \mathcal{A}, \mathbb{P})$.

Lemma 1. Let $(\alpha_k)_{k \in \mathbb{Z}}$ and $(\beta_k)_{k \in \mathbb{Z}}$ be sequences of real numbers such that $\sum_{k \in \mathbb{Z}} |\alpha_k| < \infty$ and $\sum_{k \in \mathbb{Z}} |\beta_k| < \infty$. If $X \in \mathcal{S}(\Omega, \mathcal{A}, \mathbb{P})$ then

$$F_\alpha[F_\beta[X]] = F_{\alpha\beta}[X], \quad \text{where } (\alpha\beta)_k := \sum_{j \in \mathbb{Z}} \alpha_j \beta_{k-j}.$$

Proof. Left as an exercise. □

ARMA processes

Definition 1. A time series $(X_t)_{t \in \mathbb{Z}}$ is an ARMA(p, q) process if there exist $\phi = (\phi_1, \dots, \phi_p) \in \mathbb{R}^p$ and $\theta = (\theta_1, \dots, \theta_q) \in \mathbb{R}^q$ and a white noise process Z such that

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}. \quad (1)$$

Equation (1) is to be understood “pointwise almost surely” on the underline probability space, i.e. X and Z are defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and

$$X_t(\omega) - \phi_1 X_{t-1}(\omega) - \cdots - \phi_p X_{t-p}(\omega) = Z_t(\omega) + \theta_1 Z_{t-1}(\omega) + \cdots + \theta_q Z_{t-q}(\omega)$$

for almost every $\omega \in \Omega$. The *ARMA* equation by itself does not define a unique process. A way to ensure uniqueness is to work with weakly stationary *ARMA* processes. An *ARMA*($p, 0$) process is also called an *autoregressive process* and denoted *AR*(p); an *ARMA*($0, q$) process is also called a *moving average process* and denoted *MA*(q).

For given ϕ and θ and a white noise Z , there are always many time series that satisfy the *ARMA* equation (1), but none of these may be stationary. In Example 1 in Lecture 2 we have already seen that a stationary solution of

$$X_t = \phi_1 X_{t-1} + Z_t$$

exists if and only if $|\phi_1| \neq 1$. Let $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q$. In the following theorem we shall see that a weakly stationary solution of (1) exists if the polynomial $\phi(z)$ has no roots on the unit circle $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. To see that we will use the following result.

Lemma 2. *The function $\psi(z) = \frac{\theta(z)}{\phi(z)}$ is well defined and analytic on the region $\{z \in \mathbb{C} : \phi(z) \neq 0\}$. If $\phi(z) \neq 0$ on \mathcal{S}^1 then ψ is analytic with a Laurent series representation given by*

$$\psi(z) = \sum_{j \in \mathbb{Z}} c_j z^j$$

where the series $\sum_{j \in \mathbb{Z}} c_j z^j$ is uniformly and absolutely convergent on every compact subset of the annulus $\{z \in \mathbb{C} : \delta_1 < |z| < \delta_2\}$ where $\delta_1 = \max\{|z| : z \in \mathbb{C}, |z| < 1, \phi(z) = 0\}$ and $\delta_2 = \min\{|z|, z \in \mathbb{C}, |z| > 1, \phi(z) = 0\}$.

Moreover, for every $0 < \eta < 1$ there exists $C < \infty$ such that $|c_k| \leq C\eta^{|k|}$.

If $\phi(z) \neq 0$ for $|z| \leq 1$, then $c_k = 0$ for $k < 0$ and the series $\sum_{k \in \mathbb{Z}} c_k z^k$ converges on $\{z \in \mathbb{C} : |z| < \delta_2\}$.

Proof. Admitted. □

Theorem 2. *Consider the equation*

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \quad (2)$$

where $Z \sim WN(0, \sigma^2)$ and $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are real numbers. Introduce the polynomials $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ and suppose that ϕ and θ have no common roots on $\{z \in \mathbb{C} : |z| = 1\}$. Then, Equation (6) admits a weakly stationary solution of the form

$$X_t = \sum_{k \in \mathbb{Z}} \psi_k Z_{t-k} \quad (3)$$

with $\sum_{j \in \mathbb{Z}} |\psi_j| < \infty$ if and only if $\phi(z) \neq 0$ for $|z| = 1$. In this case, the solution is unique (among any weakly stationary process) and the ψ_k 's are the coefficients of the series

$$\frac{\theta(z)}{\phi(z)} = \sum_{k \in \mathbb{Z}} \psi_k z^k$$

that is convergent on $\{z \in \mathbb{C} : \delta_1 < |z| < \delta_2\}$ where $\delta_1 = \max\{|z| : z \in \mathbb{C}, |z| < 1, \phi(z) = 0\}$ and $\delta_2 = \min\{|z| : z \in \mathbb{C}, |z| > 1, \phi(z) = 0\}$.

Remark 1. It can actually be shown that it is superfluous to assume that X be of the form (3), that is that no weakly stationary solution of (1) exists whenever ϕ has a zero on \mathcal{S}^1 .

Proof. \Leftarrow) Suppose $\phi(z) \neq 0$ for $|z| = 1$. Then, thanks to Lemma 2, there exist $\delta_1 < 1$ and $\delta_2 > 1$ such that

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = \sum_{k \in \mathbb{Z}} \psi_k z^k, \quad \delta_1 < |z| < \delta_2,$$

with $\sum_k |\psi_k| < \infty$. The process X defined as

$$X_t = \sum_{k \in \mathbb{Z}} \psi_k Z_{t-k} = (F_\psi[Z])_t, \quad \forall t \in \mathbb{Z}$$

is a weakly stationary process (see Theorem (1)) and a solution of (6). Indeed, thanks to Lemma 1, $F_\phi[X] = F_\phi[F_\psi[Z]] = F_{\phi\psi}[Z] = F_\theta[Z]$.

Uniqueness: Let X be a weakly stationary process solution of (6), then

$$F_\phi[X] = F_\theta[Z]. \quad (4)$$

Since $\phi(z) \neq 0$ on \mathcal{S}^1 , thanks to Lemma 2 there exist $\delta_1 < 1$ and $\delta_2 > 1$ such that

$$\xi(z) = \frac{1}{\phi(z)} = \sum_{k \in \mathbb{Z}} \xi_k z^k, \quad \delta_1 < |z| < \delta_2$$

with $\sum_{k \in \mathbb{Z}} |\xi_k| < \infty$. We can then apply the operator F_ξ in (4) getting

$$X = F_\xi[F_\phi[X]] = F_\xi[F_\theta[Z]] = F_\psi[Z].$$

\Rightarrow) Let us now show that if X is a weakly stationary process such that $F_\phi[X] = F_\theta[Z]$ and of the form $X_t = \sum_{k \in \mathbb{Z}} \eta_k Z_{t-k}$ for some $(\eta_k)_{k \in \mathbb{Z}}$ such that $\sum_{k \in \mathbb{Z}} |\eta_k| < \infty$, then $\phi(z) \neq 0$ on \mathcal{S}^1 . Since X is solution of (1) $F_\phi[F_\eta[Z]] = F_\theta[Z]$, therefore, using Lemma 1, $F_{\phi\eta}[Z] = F_\theta[Z]$. Let $\beta_k = \sum_{j \in \mathbb{Z}} \phi_j \eta_{k-j}$; then, from $\sum_{k \in \mathbb{Z}} \beta_k Z_{t-k} = \sum_{j=0}^q \theta_j Z_{t-j}$, (with the convention $\theta_0 = 1$) it follows $\sum_{k \in \mathbb{Z}} \beta_k \mathbb{E}[Z_{t-k} Z_{t-l}] = \sum_{j=1}^q \theta_j \mathbb{E}[Z_{t-j} Z_{t-l}]$ for all l and thus $\beta_l = \theta_l$, $l = 0, \dots, q$, and $\beta_l = 0$ otherwise. Set $\eta(z) = \sum_{k \in \mathbb{Z}} \eta_k z^k$ and observe that $\beta(z) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^p \phi_j \eta_{k-j} z^k = \sum_{j=0}^p \phi_j z^j \sum_{k \in \mathbb{Z}} \eta_k z^k$, i.e. formally $\theta(z) = \phi(z)\eta(z)$. Because $|\eta(z)| \leq \sum_{k \in \mathbb{Z}} |\eta_k| < \infty$ on \mathcal{S}^1 and θ and ϕ have no common roots on \mathcal{S}^1 then $\phi(z)$ must be different from 0 on \mathcal{S}^1 . \square

Example 1. Consider

$$X_t = \phi_1 X_{t-1} + Z_t, \quad t \in \mathbb{Z}. \quad (5)$$

Then, $\phi(z) = 1 - \phi_1 z$, $\theta(z) = 1$ and $\psi(z) = \frac{1}{1 - \phi_1 z}$. In particular, $\psi(z) \neq 0$ on \mathcal{S}^1 if and only if $|\phi_1| \neq 1$.

- If $|\phi_1| < 1$ then $\delta_1 = \max\{|z| : |z| < 1, z = \frac{1}{\phi_1}\} = \max\{\emptyset\} = -\infty$ and $\delta_2 = \frac{1}{|\phi_1|}$. On $\{z \in \mathbb{C} : |z| < \frac{1}{|\phi_1|}\}$ we have

$$\frac{\theta(z)}{\phi(z)} = \frac{1}{1 - \phi_1 z} = \sum_{k=0}^{\infty} \phi_1^k z^k = \sum_{k \in \mathbb{Z}} \psi_k z^k$$

with

$$\psi_k = \begin{cases} \phi_1^k & \text{if } k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the weakly stationary solution of (5) is

$$X_t = \sum_{k=0}^{\infty} \phi_1^k Z_{t-k}.$$

- If $|\phi_1| > 1$ then $\delta_1 = \frac{1}{|\phi_1|}$ and $\delta_2 = \min\{\emptyset\} = +\infty$. On $\{z \in \mathbb{C} : |z| > \frac{1}{|\phi_1|}\}$ we have

$$\frac{1}{1 - \phi_1 z} = \frac{-(\phi_1 z)^{-1}}{1 - (\phi_1 z)^{-1}} = -(\phi_1 z)^{-1} \sum_{k=0}^{\infty} (\phi_1 z)^{-k} = - \sum_{l \geq 1} \phi_1^{-l} z^{-l} = - \sum_{k \leq -1} \phi_1^k z^k = \sum_{k \in \mathbb{Z}} \psi_k z^k$$

with

$$\psi_k = \begin{cases} -\phi_1^k & \text{if } k \leq -1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the weakly stationary solution of (5) is

$$X_t = - \sum_{k \leq -1} \phi_1^k Z_{t-k} = - \sum_{k \geq 1} \phi_1^{-k} Z_{t+k}.$$

To sum up:

- If $|\phi_1| = 1$ there are no weakly stationary solution of (5),
- If $|\phi_1| < 1$ the weakly stationary solution of (5) is $X_t = \sum_{j \geq 0} \phi_1^j Z_{t-j}$,
- If $|\phi_1| > 1$ the weakly stationary solution of (5) is $X_t = - \sum_{j \geq 1} \phi_1^{-j} Z_{t+j}$,

as already seen in Lecture 2.

Definition 2. A weakly stationary ARMA(p, q)-process is called causal if $X_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k}$, $t \in \mathbb{Z}$, $Z \sim WN(0, \sigma^2)$ and $\sum_{k \in \mathbb{Z}} |\psi_k| < \infty$.

Theorem 3. Let X be a weakly stationary ARMA(p, q)-process on \mathbb{Z} :

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q} \quad (6)$$

where $Z \sim WN(0, \sigma^2)$ and $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are real numbers. Introduce the polynomials $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$ and suppose that ϕ and θ have no common roots on $\{z \in \mathbb{C} : |z| \leq 1\}$. Then X is causal if and only if $\phi(z) \neq 0$ for $z \in \mathbb{C}$ with $|z| \leq 1$. In that case $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ holds where $\psi(z) := \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}$ for $|z| \leq 1$.

Proof. Left as an exercise. \square