Lecture 2

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The most elementary (non trivial) weakly stationary process is the *white noise*.

Definition 1. A weakly stationary process $Z = (Z_t)_{t\in\mathbb{Z}}$ with mean 0 and autocovariance function

$$c(h) = \begin{cases} \sigma^2 & h = 0, \\ 0 & h \neq 0 \end{cases}$$

is called *white noise* $Z \sim WN(0, \sigma^2)$. If $Z = (Z_t)_{t\in\mathbb{Z}}$ is even an i.i.d. sequence and $Z \sim WN(0, \sigma^2)$ we write $Z \sim IID(0, \sigma^2)$.

Remark 1. White noise processes and i.i.d. sequences are not equivalent notions.

One of the simplest ways to construct a time series X that is strictly stationary is to "filter" an i.i.d. sequence of random variables, that is to take a sequence of i.i.d. random variables $(Z_t)_{t\in\mathbb{Z}}$ and define

$$X_t = g(Z_t, Z_{t-1}, \dots, Z_{t-q})$$

for some real-valued function g. Then X is strictly stationary and q-dependent i.e. X_t and X_s are independent whenever $|t - s| > q$. We say that a weakly stationary time series is q-correlated if $c(k) = 0$ for all $|k| > q$.

Definition 2. $X = (X_t)_{t\in\mathbb{Z}}$ is said to be a *moving average of order* q ($MA(q)$) if

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \tag{1}$$

where $Z \sim WN(0, \sigma^2)$ and $\theta_1, \dots, \theta_q \in \mathbb{R}$.

Remark 2. Clearly, if X is a $MA(q)$ then X is weakly stationary and q-correlated. The inverse is also true: If X is a weakly stationary q-correlated time series with mean 0 then X is a $MA(q)$. (See e.g. [1].)

Definition 3. X is a *linear process* if it has the representation

$$X_t = \sum_{j\in\mathbb{Z}} \psi_j Z_{t-j} \tag{2}$$

for all t, where $Z \sim WN(0, \sigma^2)$ and $(\psi_j)_{j\in\mathbb{Z}}$ is a sequence of real numbers such that $\sum_{j\in\mathbb{Z}} |\psi_j| < \infty$. A linear process is called a *moving average* or $MA(\infty)$ if $\psi_j = 0\ \forall j < 0$, i.e. $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$.

Several questions come naturally in connection with the representation (2) like: How to interpret an infinite sum of random variables? Is X well defined?

Theorem 1. *If $(Z_t)_{t\in\mathbb{Z}}$ and $(\psi_j)_{j\in\mathbb{Z}}$ are any sequences such that*

$$\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|] < \infty \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |\psi_j| < \infty$$

then, for all $t \in \mathbb{Z}$, the sequence

$$X_{n,t} := \sum_{k=-n}^{n} \psi_k Z_{t-k}$$

converges almost surely to a limit denoted by

$$X_t := \sum_{k\in\mathbb{Z}} \psi_k Z_{t-k}.$$

Also, $\mathbb{E}[|X_t|] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[|X_{n,t} - X_t|] = 0$. If in addition $\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|^2] < \infty$, then $\mathbb{E}[|X_t|^2] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[|X_{n,t} - X_t|^2] = 0$.

Proof. $\forall t \in \mathbb{Z}, n \in \mathbb{N}$ set $U_{n,t} = \sum_{k=-n}^{n} |\psi_k||Z_{t-k}|$. In particular, $(U_{n,t})_{n\geq 0}$ is an increasing sequence of positive random variables and converges point-wise to $\sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}|$. By means of Beppo Levi theorem we conclude that

$$\lim_{n\to\infty} \mathbb{E}[U_{n,t}] = \mathbb{E}\Bigg[\sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}|\Bigg].$$

Furthermore, observe that

$$\mathbb{E}[U_{n,t}] = \sum_{k=-n}^{n} |\psi_k| \mathbb{E}[|Z_{t-k}|] \leq \Big(\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|]\Big) \sum_{k=-n}^{n} |\psi_k| < \infty.$$

Therefore,

$$\mathbb{E}\Bigg[\sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}|\Bigg] \leq \Big(\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|]\Big) \sum_{k\in\mathbb{Z}} |\psi_k| < \infty.$$

It follows that there exists $\Omega_0 \in \mathcal{A}$ such that $\mathbb{P}(\Omega_0) = 1$ and

$$\sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}(\omega)| < \infty \quad \forall \omega \in \Omega_0,$$

i.e. for all $\omega \in \Omega_0$

$$|X_{n,t}(\omega) - X_t(\omega)| \leq \sum_{|k|>n} |\psi_k||Z_{t-k}(\omega)| \xrightarrow[n\to\infty]{} 0,$$

that is $(X_{n,t})_{n\in\mathbb{N}}$ converges to X_t almost surely and, by means of Fatou lemma,

$$\mathbb{E}[|X_t|] = \mathbb{E}\big[\lim_n |X_{n,t}|\big] \leq \liminf_n \mathbb{E}[|X_{n,t}|] \leq \liminf_n \mathbb{E}[U_{n,t}] < \infty.$$

Moreover, we also have

$$|X_{n,t} - X_t| \leq \sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}| \quad \text{and} \quad \mathbb{E}\Big[\sum_{k\in\mathbb{Z}} |\psi_k||Z_{t-k}|\Big] < \infty.$$

Hence, by dominated convergence theorem, $(X_{n,t})_{n\in\mathbb{N}}$ converges in L_1 to X_t.

Suppose now that $\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|^2] < \infty$. Observe that $(X_{m,t})_{m\geq 0}$ is a Cauchy sequence in $L_2(\Omega, \mathcal{A}, \mathbb{P})$. Indeed, for all $p \geq q$, we have

$$\begin{aligned}
\|X_{p,t} - X_{q,t}\|_2^2 &= \Big\| \sum_{q<|k|<p} \psi_k Z_{t-k} \Big\|_2^2 \\
&= \sum_{q<|k|<p} \sum_{q<|j|<p} \psi_j \psi_k \mathbb{E}[Z_{t-j} Z_{t-k}] \\
&\leq \sup_t \mathbb{E}[|Z_t|^2] \Big(\sum_{q<|j|<p} |\psi_j| \Big)^2 \xrightarrow[p,q\to\infty]{} 0.
\end{aligned}$$

Since $L_2(\Omega, \mathcal{A}, \mathbb{P})$ is complete, $(X_{n,t})_{n\in\mathbb{N}}$ converges in L_2 to some random variable X_t^* as $n \to \infty$. By means of Fatou lemma we then get

$$\mathbb{E}[|X_t - X_t^*|^2] = \mathbb{E}[\lim_n |X_{n,t} - X_t^*|^2] \leq \liminf_n \mathbb{E}[|X_{n,t} - X_t^*|^2] = 0.$$

Therefore, $\mathbb{E}[|X_t|^2] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[|X_{n,t} - X_t|^2] = 0$.

□

Remark 3. Observe that $\lim_{n\to\infty} \mathbb{E}[|X_{n,t} - X_t|] = 0$ implies $\mathbb{E}[X_t] = \lim_{n\to\infty} \mathbb{E}[X_{n,t}]$. Moreover, using the continuity of the scalar product in L_2 (i.e. if $\|U_n - U\|_2 \to 0$ and $\|V_n - V\|_2 \to 0$ then $\mathbb{E}[U_n V_n] \to \mathbb{E}[UV]$) and under the assumption $\sup_{t\in\mathbb{Z}} \mathbb{E}[|Z_t|^2] < \infty$, we also have

$$\mathbb{E}[X_t X_{t+h}] = \lim_{n\to\infty} \sum_{k=-n}^{n} \sum_{j=-n}^{n} \psi_k \psi_j \mathbb{E}[Z_{t+h-k} Z_{t-j}].$$

Theorem 2. *A linear process $X = (X_t)_{t\in\mathbb{Z}}$ with representation (2) is weakly stationary with mean 0 and autocovariance function*

$$c(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2.$$

Proof. Using that $Z \sim WN(0, \sigma^2)$ and Remark 3 we have:

$$\mathbb{E}[X_t] = \lim_{n\to\infty} \mathbb{E}\Big[\sum_{k=-n}^{n} \psi_k Z_{t-k}\Big] = 0$$

and

$$\begin{aligned} c(h) = \mathbb{E}[X_t X_{t+h}] &= \lim_{n\to\infty} \sum_{k=-n}^{n} \sum_{j=-n}^{n} \psi_k \psi_j \mathbb{E}[Z_{t+h-k} Z_{t-j}] \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \psi_k \psi_j \mathbb{E}[Z_{j+h-k} Z_0] \\ &= \sum_{k'\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \psi_{k'+h} \psi_j \mathbb{E}[Z_{j-k'} Z_0] \\ &= \sum_{j\in\mathbb{Z}} \psi_{h+j} \psi_j \mathbb{E}[Z_0^2]. \end{aligned}$$

□

Corollary 1. *Let X be a $MA(q)$ with the same representation as in (1), with the convention $\theta_0 = 1$. Then X is a weakly stationary process with mean 0 and autocovariance function*

$$c(h) = \begin{cases} \sigma^2 \sum_{k=0}^{q-|h|} \theta_k \theta_{k+|h|} & \text{if } |h| \le q, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. The process $X = (X_t)_{t\in\mathbb{Z}}$ is said to be an *autoregressive process of order p*, $AR(p)$, if it is weakly stationary and it is solution of

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t \tag{3}$$

where $Z \sim WN(0, \sigma^2)$ and $\phi_1, \ldots, \phi_p$ are real numbers.

Remark 4. X solution of (3) is said autoregressive since the equation (3) has the form of a regression of the p previous values plus the noise.

Example 1. Given $Z \sim WN(0, \sigma^2)$ and a number θ, consider the equation

$$X_t = \theta X_{t-1} + Z_t, \quad t \in \mathbb{Z}. \tag{4}$$

To make the statement (4) formally correct, we fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and look for a solution $X_t(\omega) = \theta X_{t-1}(\omega) + Z_t(\omega)$ for (almost) every $\omega \in \Omega$. The autoregressive equation (4) does not uniquely determine X: a trivial solution is to define Z and X_0 in some arbitrary way on $(\Omega, \mathcal{A}, \mathbb{P})$ and next define the remaining variables X_t for $t \in \mathbb{Z} \setminus \{0\}$ by the equation $X_t = \theta X_{t-1} + Z_t$. If, however, we insist that X be stationary, then the solution is unique, whenever it exists. The proof of the uniqueness will be part of next lecture. Here we investigate existence of a stationary solution depending on θ.

- If $|\theta| < 1$: By iteration we find

$$\begin{aligned} X_t &= \theta(\theta X_{t-2} + Z_{t-1}) + Z_t = \theta^2 X_{t-2} + \theta Z_{t-1} + Z_t = \ldots \\ &= \theta^k X_{t-k} + \theta^{k-1} Z_{t-k+1} + \cdots + \theta Z_{t-1} + Z_t. \end{aligned}$$

Taking $k \to \infty$ we have that a good candidate for the solution is

$$\sum_{j=0}^{\infty} \theta^j Z_{t-j}. \tag{5}$$

(5) is a linear process and hence weakly stationary with mean 0 and autocovariance function

$$c(h) = \begin{cases} \sigma^2 \frac{\theta^h}{1-\theta^2} & h \geq 0, \\ \frac{\sigma^2 \theta^{-h}}{1-\theta^2} & h < 0. \end{cases}$$

By substitution it is immediate to see that $\sum_{j=0}^{\infty} \theta^j Z_{t-j}$ is a solution of (5) since

$$\sum_{j=0}^{\infty} \theta^j Z_{t-j} = \theta \sum_{j=0}^{\infty} \theta^j Z_{t-1-j} + Z_t.$$

- If $|\theta| > 1$ we know that

  $$X_t = \theta^{k+1} X_{t-k-1} + \theta^k Z_{t-k} + \cdots + \theta^2 Z_{t-2} + \theta Z_{t-1} + Z_t.$$

  Hence,

  $$\mathbb{E}\left[\left| X_t - \sum_{j=0}^{k} \theta^j Z_{t-j} \right|^2\right] = \mathbb{E}\left[(\theta^{k+1} X_{t-k-1})^2\right] = \theta^{2k+2} \mathbb{E}\left[X_{t-k-1}^2\right]$$

  diverges for $k \to \infty$.

  Nevertheless we can write

  $$X_t = -\frac{Z_{t+1}}{\theta} + \frac{X_{t+1}}{\theta}.$$

  By iteration

  $$\begin{aligned} X_t &= -\frac{Z_{t+1}}{\theta} - \frac{Z_{t+2}}{\theta^2} + \frac{X_{t+2}}{\theta^2} = \dots \\ &= -\frac{Z_{t+1}}{\theta} - \frac{Z_{t+2}}{\theta^2} - \cdots - \frac{Z_{t+k+1}}{\theta^{k+1}} + \frac{X_{t+k+1}}{\theta^{k+1}}. \end{aligned}$$

  As before we conclude that

  $$X_t = -\sum_{j \geq 1} \frac{Z_{t+j}}{\theta^j}. \tag{6}$$

  Observe that (6) is not a *causal* series since the series at time t only depends on its future.

- If $|\theta| = 1$. Let us suppose that there exists a weakly stationary solution with mean μ and autocovariance $c(\cdot)$. Then

  $$\mathbb{E}\left[\left| X_t - \sum_{j=0}^{k-1} \theta^j Z_{t-j} \right|^2\right] = \mathbb{E}\left[|\theta^k X_{t-k}|^2\right] = |\theta|^{2k} \mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[X_t^2\right]. \tag{7}$$

We also have

$$\mathbb{E}\left[\left|X_t - \sum_{j=0}^{k-1} \theta^j Z_{t-j}\right|^2\right] = \mathbb{E}\left[X_t^2\right] - 2\mathbb{E}\left[X_t \sum_{j=0}^{k-1} \theta^j Z_{t-j}\right] + \mathbb{E}\left[\left(\sum_{j=0}^{k-1} \theta^j Z_{t-j}\right)^2\right] \quad (8)$$

and hence, from (7) and (8), we deduce

$$\mathbb{E}\left[\left(\sum_{j=0}^{k-1} \theta^j Z_{t-j}\right)^2\right] = 2\mathbb{E}\left[X_t \sum_{j=0}^{k-1} \theta^j Z_{t-j}\right].$$

Moreover, using that $Z \sim WN(0, \sigma^2)$, we get

$$\mathbb{E}\left[\left(\sum_{j=0}^{k-1} \theta^j Z_{t-j}\right)^2\right] = \sum_{j=0}^{k-1} \theta^{2j} \mathbb{E}[Z_{t-j}^2] = k\sigma^2.$$

Therefore, by Cauchy-Schwarz, we obtain

$$k\sigma^2 = 2\mathbb{E}\left[X_t \sum_{j=0}^{k-1} \theta^j Z_{t-j}\right] \leq 2\|X_t\|_2 \left\|\sum_{j=0}^{k-1} \theta^j Z_{t-j}\right\|_2 \leq 2\sqrt{c(0) + \mu^2}\sqrt{k}\sigma,$$

which is absurd for k big enough.

To sum up:

- If $|\theta| = 1$ there are no stationary solution of (5).
- If $|\theta| < 1$ a solution of (5) is $X_t = \sum_{j \geq 0} \theta^j Z_{t-j}$.
- If $|\theta| > 1$ a solution of (5) is $X_t = -\sum_{j \geq 1} \theta^{-j} Z_{t+j}$.

Definition 5. A filter with coefficients ψ_j is *causal* if $\psi_j = 0$ for all $j < 0$.

For a causal filter the variable

$$Y_t = \sum_{j \in \mathbb{Z}} \psi_j X_{t-j}$$

depends only on the values X_t, X_{t-1}, ... of the original time series in the present and past, not the future!

References

[1] Brockwell, P. and Davis, R. Time Series: Theory and Methods, Springer, 2006.