Lecture 10

Inference for jumps Lévy processes from high frequency observations: a direct approach

By now, we have seen a way to estimate the Lévy measure of a compound Poisson process. Let us now see how to estimate the Lévy measure of a general Lévy process by means of a compound Poisson process approximation. More precisely, we will consider the following setting. Let X be a pure jumps Lévy process with Lévy measure ν. If $\int_{|x|\leq 1}|x|\nu(dx)<\infty$, the process associated with the sum of all the jumps of X is a Lévy process with Lévy triplet given by $(\int_{|x|\leq 1}x\nu(dx),0,\nu)$. In the following, if $\int_{|x|\leq 1}|x|\nu(dx)<\infty$ we will assume that

$$X_t := \sum_{s\leq t}\Delta X_s \mathbf{1}_{\Delta X_s\neq 0}, \tag{1}$$

otherwise we will consider Lévy processes with Lévy triplets $(0,0,\nu)$. That is, we will focus on the class $\mathscr{L}$ of Lévy processes with Lévy triplet $(\gamma_\nu,0,\nu)$ where

$$\gamma_\nu := \begin{cases}\int_{|x|\leq 1}x\nu(dx) & \text{if } \int_{|x|\leq 1}|x|\nu(dx)<\infty\\ 0 & \text{otherwise.}\end{cases}$$

Thanks to the Lévy-Itô decomposition any element X of $\mathscr{L}$ can be written as follows: for any $0<\varepsilon\leq 1$

$$X_t = tb_\nu(\varepsilon)+M_t(\varepsilon)+B_t(\varepsilon),$$

where:

- the drift is given by

$$b_\nu(\varepsilon) := \begin{cases}\int_{|x|\leq \varepsilon}x\nu(dx) & \text{if } \int_{|x|\leq 1}|x|\nu(dx)<\infty\\ -\int_{\varepsilon<|x|\leq 1}x\nu(dx) & \text{otherwise.}\end{cases}$$

- $M(\varepsilon)=(M_t(\varepsilon))_{t\geq 0}$ is the martingale associated with the jumps of X the magnitude of which is less then ε. Formally:

$$M_t(\varepsilon)=\lim_{\eta\to 0}\bigg(\sum_{s\leq t}\Delta X_s\mathbf{1}_{\eta<|\Delta X_s|\leq\varepsilon}-t\int_{\eta<|x|\leq\varepsilon}x\nu(dx)\bigg).$$

- $B(\varepsilon) = (B_t(\varepsilon))_{t\geq 0}$ is the Lévy process associated with the jumps of X the magnitude of which is bigger then ε. Formally:

$$B_t(\varepsilon) = \sum_{s\leq t} \Delta X_s \mathbf{1}_{|\Delta X_s|>\varepsilon}.$$

  In particular, $B(\varepsilon)$ is a compound Poisson process with intensity $\lambda_\varepsilon := \nu(\mathbb{R}\setminus[-\varepsilon,\varepsilon])$ and jump distribution $F_\varepsilon(dx) := \frac{\nu(dx)}{\lambda_\varepsilon}\mathbf{1}_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}$.

Let us suppose that ν and F_ε are absolutely continuous with respect to the Lebesgue measure with a density given by ρ and f_ε, respectively. Our aim is to estimate ρ and let us say that we are interested in finding an estimator $\widehat{\rho}$ of ρ on a compact set A uniformly bounded away from zero that "works well" with respect to the L_p-risk:

$$\mathbb{E}\bigg[\int_A |\widehat{\rho}(x)-\rho(x)|^p dx\bigg].$$

Clearly, for ε small enough,

$$\rho(x)\mathbf{1}_A(x) = \lambda_\varepsilon f_\varepsilon(x)\mathbf{1}_{|x|>\varepsilon}\mathbf{1}_A(x).$$

Thus, an estimator of ρ on A is given by

$$\widehat{\rho}(x) = \widehat{\lambda}_\varepsilon \widehat{f}_\varepsilon(x) \quad \forall x\in A,$$

where $\widehat{\lambda}_\varepsilon$ and $\widehat{f}_\varepsilon$ are estimators of λ_ε and f_ε, respectively. Observe that:

$$\begin{aligned}\mathbb{E}\bigg[\int_A |\widehat{\rho}(x)-\rho(x)|^p dx\bigg] &= \int_A \mathbb{E}[|\widehat{\lambda}_\varepsilon \widehat{f}_\varepsilon(x) - \lambda_\varepsilon f_\varepsilon(x)|^p]dx \\ &\leq 2^{p-1}\mathbb{E}[|\widehat{\lambda}_\varepsilon - \lambda_\varepsilon|^p]\int_A |f_\varepsilon(x)|^p dx + 2^{p-1}\mathbb{E}\bigg[|\widehat{\lambda}_\varepsilon|^p \int_A |\widehat{f}_\varepsilon(x) - f(x)|^p dx\bigg].\end{aligned}$$

Also,

$$\int_A \mathbb{E}\big[|\widehat{\lambda}_\varepsilon|^p |\widehat{f}_\varepsilon(x) - f_\varepsilon(x)|^p\big]dx \leq \sqrt{\mathbb{E}\big[|\widehat{\lambda}_\varepsilon|^{2p}\big]}\int_A \sqrt{\mathbb{E}\big[|\widehat{f}_\varepsilon(x) - f_\varepsilon(x)|^{2p}\big]dx}.$$

In particular, in order to control the L_p-risk of $\widehat{\rho}$ it is enough to separately control the L_p-risks of $\widehat{\lambda}_\varepsilon$ and $\widehat{f}_\varepsilon$. We will only focus on the estimation of λ_ε, the estimation of f_ε being similar to the estimation of f in Lecture 9 but more involved due to the presence of the small jumps (see, e.g., [3]).

Estimation of λ_ε

In Lecture 8 we have seen that, for any Lévy process X with Lévy measure ν, we have:

$$\lim_{t\to 0}\frac{\mathbb{P}(|X_t|>\varepsilon)}{t} = \nu(\mathbb{R}\setminus[-\varepsilon,\varepsilon]);$$

that is, using the previous notation,

$$\lim_{t\to 0} \frac{\mathbb{P}(|X_t| > \varepsilon)}{t} = \lambda_\varepsilon. \tag{2}$$

Equation (2) joined with the fact that we are assuming to have high frequency observations of X, justifies the following definition

$$\widehat{\lambda}_\varepsilon := \frac{n(\varepsilon)}{n\Delta} \tag{3}$$

where $n(\varepsilon) = \sum_{i=1}^n \mathbf{1}_{|X_{i\Delta} - X_{(i-1)\Delta}| > \varepsilon}$.

Let us now compute the L_p-risk of $\widehat{\lambda}_\varepsilon$. To that aim, we will apply Rosenthal inequality, that is we will use the following result.

Rosenthal inequality: Let $\xi_1, \dots, \xi_n$ be independent random variables with $\mathbb{E}[\xi_i] = 0$, $\mathbb{E}\big[|\xi_i|^p\big] < \infty$ for $p > 2$, $i = 1, \dots, n$. Let $S_n = \sum_{i=1}^n \xi_i$. Then, there exists $0 < C < \infty$ such that

$$\mathbb{E}[|S_n|^p] \le C \max\bigg(\sum_{i=1}^n \mathbb{E}[|\xi_i|^p], \Big(\sum_{i=1}^n \mathbb{E}[\xi_i^2]\Big)^{p/2}\bigg).$$

Using the convexity inequality $(a+b)^p \le 2^{p-1}a^p + 2^{p-1}b^p$ we have:

$$\begin{aligned}
\mathbb{E}[|\lambda_\varepsilon - \widehat{\lambda}_\varepsilon|^p] &= \mathbb{E}\bigg[\Big|\lambda_\varepsilon - \mathbb{E}\Big[\frac{n(\varepsilon)}{n\Delta}\Big] + \mathbb{E}\Big[\frac{n(\varepsilon)}{n\Delta}\Big] - \frac{n(\varepsilon)}{n\Delta}\Big|^p\bigg] \\
&\le 2^{p-1}\Big|\lambda_\varepsilon - \frac{\mathbb{P}(|X_\Delta| > \varepsilon)}{\Delta}\Big|^p + 2^{p-1}\frac{1}{\Delta^p}\mathbb{E}\bigg[\Big|\mathbb{P}(|X_\Delta| > \varepsilon) - \frac{n(\varepsilon)}{n}\Big|^p\bigg].
\end{aligned}$$

Set

$$U_i := \frac{\mathbf{1}_{(\varepsilon,\infty)}(|X_{i\Delta} - X_{(i-1)\Delta}|) - \mathbb{P}(|X_\Delta| > \varepsilon)}{n}, \quad i = 1, \dots, n.$$

In particular, $(U_i)_{i=1}^n$ are i.i.d. centered random variables such that

$$\Big|\sum_{i=1}^n U_i\Big| = \Big|\mathbb{P}(|X_\Delta| > \varepsilon) - \frac{n(\varepsilon)}{n}\Big|.$$

Therefore, applying Rosenthal inequality for $p > 2$, we get

$$\mathbb{E}\bigg[\Big|\mathbb{P}(|X_\Delta| > \varepsilon) - \frac{n(\varepsilon)}{n}\Big|^p\bigg] \le C \max\bigg(\sum_{i=1}^n \mathbb{E}[|U_i|^p], \Big(\sum_{i=1}^n \mathbb{E}[U_i^2]\Big)^{p/2}\bigg),$$

for some positive constant C.

Observe that

$$\mathbb{E}[U_1^2] = \frac{\mathbb{P}(|X_\Delta| > \varepsilon)(1 - \mathbb{P}(|X_\Delta| > \varepsilon))}{n^2} \le \frac{\mathbb{P}(|X_\Delta| > \varepsilon)}{n^2}$$

and so we conclude that

$$\Big(\sum_{i=1}^n \mathbb{E}[U_i^2]\Big)^{p/2} \le \Big(\frac{\mathbb{P}(|X_\Delta| > \varepsilon)}{n}\Big)^{p/2}.$$

Moreover, since

$$\begin{aligned}\mathbb{E}\Big[|\mathbf{1}_{|X_\Delta|>\varepsilon}-\mathbb{P}(|X_\Delta|>\varepsilon)|^p\Big]&=\mathbb{E}\Big[\big|\mathbf{1}_{|X_\Delta|>\varepsilon}-\mathbb{P}(|X_\Delta|>\varepsilon)\big|^2\big|\mathbf{1}_{|X_\Delta|>\varepsilon}-\mathbb{P}(|X_\Delta|>\varepsilon)\big|^{p-2}\Big]\\&\leq\mathbb{E}\Big[\big|\mathbf{1}_{|X_\Delta|>\varepsilon}-\mathbb{P}(|X_\Delta|>\varepsilon)\big|^2\Big]\leq\mathbb{P}(|X_\Delta|>\varepsilon),\end{aligned}$$

we obtain that $\mathbb{E}[|U_1|^p]\leq\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n^p}$ and thus, for $p>2$, there exists a contant C (only dependent on p) such that:

$$\mathbb{E}\Big[\Big|\mathbb{P}(|X_\Delta|>\varepsilon)-\frac{n(\varepsilon)}{n}\Big|^p\Big]\leq C\Big(n^{p-1}\mathbb{P}(|X_\Delta|>\varepsilon),\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n}\Big)^{p/2}\Big).$$

Hence, for $p>2$:

$$\mathbb{E}[|\lambda_\varepsilon-\widehat{\lambda}_\varepsilon|^p]\leq 2^{p-1}\Big|\lambda_\varepsilon-\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}\Big|^p+2^{p-1}\frac{C}{\Delta^p}\max\Big(n^{p-1}\mathbb{P}(|X_\Delta|>\varepsilon),\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n}\Big)^{p/2}\Big).$$

Let us now study the case $p=2$:

$$\mathbb{E}[|\lambda_\varepsilon-\widehat{\lambda}_\varepsilon|^2]=\big(\lambda_\varepsilon-\mathbb{E}[\widehat{\lambda}_\varepsilon]\big)^2+\mathrm{Var}(\widehat{\lambda}_\varepsilon)=\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}-\lambda_\varepsilon\Big)^2+\frac{\mathbb{P}(|X_\Delta|>\varepsilon)(1-\mathbb{P}(|X_\Delta|>\varepsilon))}{n\Delta^2},$$

i.e.

$$\mathbb{E}[|\lambda_\varepsilon-\widehat{\lambda}_\varepsilon|^2]\leq\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}-\lambda_\varepsilon\Big)^2+\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n\Delta^2}. \tag{4}$$

If $1\leq p<2$, using Jensen inequality together with (4), we get:

$$\begin{aligned}\mathbb{E}\big[|\lambda_\varepsilon-\widehat{\lambda}_\varepsilon|^p\big]&\leq\Big(\mathbb{E}\big[(\lambda_\varepsilon-\widehat{\lambda}_\varepsilon)^2\big]\Big)^{p/2}\\&\leq\bigg(\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}-\lambda_\varepsilon\Big)^2+\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n\Delta^2}\bigg)^{p/2}\\&\leq\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}-\lambda_\varepsilon\Big)^p+\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n\Delta^2}\Big)^{p/2}.\end{aligned}$$

Moreover, for any $\varepsilon=\varepsilon_n\in(0,1]$ if $n\mathbb{P}(|X_\Delta|>\varepsilon_n)\to\infty$ as $n\to\infty$, it can be proven (for instance, by means of Bernstein inequality) that, for n large enough

$$\mathbb{E}\big[|\widehat{\lambda}_\varepsilon-\lambda_\varepsilon|^p\big]\leq\Big|\lambda_\varepsilon-\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}\Big|^p+O\bigg(\Big(\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{n\Delta^2}\Big)^{\frac{p}{2}}\bigg),\quad\forall p\in[1,\infty).$$

Remark 1. *In general, the quantity* $\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}$ *is not easy to handle. From Lecture 8 we know that, for all* $\varepsilon>0$,

$$\lim_{\Delta\to0}\Big|\lambda_\varepsilon-\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}\Big|=0$$

but the rate of convergence is not known in general. Nevertheless, one can notice that in many cases of interest, in terms of $\lambda_\varepsilon=\int_{|x|>\varepsilon}\nu(dx)$,

$$\Big|\lambda_\varepsilon-\frac{\mathbb{P}(|X_\Delta|>\varepsilon)}{\Delta}\Big|=O(\Delta\lambda_\varepsilon^2),\qquad\textit{as long as}\qquad\Delta\lambda_\varepsilon\to0.$$

In this case the rate of convergence of $\widehat{\lambda}_\varepsilon$ *in* L_p *risk,* $p\in[1,\infty)$, *is given by:*

$$\mathbb{E}\big[|\widehat{\lambda}_\varepsilon-\lambda_\varepsilon|^p\big]=O\Big(\Big(\frac{\lambda_\varepsilon+\lambda_\varepsilon^2\Delta}{n\Delta}\Big)^{\frac{p}{2}}+\lambda_\varepsilon^{2p}\Delta^p\Big).$$

Compound Poisson approximation

Theorem 1. *Any infinitely divisible probability distribution can be obtained as the limit of a sequence of compound Poisson distributions.*

Before proving Theorem 1 let us recall some facts regarding the characteristic functions of infinitely divisible distributions. See, e.g., [6] for a proof.

Lemma 1. *If μ is an infinitely divisible distribution, then its characteristic function ϕ_μ has no zero, i.e. $\phi_\mu(u) \neq 0$ for any $u \in \mathbb{R}$.*

Lemma 2. *Suppose that $\phi : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\phi(0) = 1$ and $\phi(z) \neq 0$ for any z. Then, there exists a unique continuous function $f : \mathbb{R} \to \mathbb{C}$ such that $f(0) = 0$ and $e^{f(z)} = \phi(z)$. Moreover, for any positive integer n there exists a unique continuous function $g_n : \mathbb{R} \to \mathbb{C}$ such that $g_n(0) = 1$ and $g_n(z)^n = \phi(z)$. They are related as $g_n(z) = e^{f(z)/n}$.*

In the following, we will write $f(z) = \log(\phi(z))$ and $g_n(z) = \phi(z)^{1/n}$.

Proof of Theorem 1. Let ϕ be the characteristic function of an arbitrary infinitely divisible probability measure μ. Since μ is infinitely divisible, for any $n \geq 1$ there exists a probability measure $\mu^{1/n}$ such that μ can be written as the convolution of n-times $\mu^{1/n}$:

$$\mu = \mu^{1/n} * \cdots * \mu^{1/n}. \tag{5}$$

Let us denote by $\phi^{1/n}$ the characteristic function of $\mu^{1/n}$. Then, (5) is equivalent to

$$\phi(u) = \Big(\phi^{1/n}(u)\Big)^n.$$

Define

$$\phi_n(u) := \exp\Big(n\big(\phi^{1/n}(u) - 1\big)\Big) = \exp\Big(\int_{\mathbb{R}} (e^{iux} - 1) n\mu^{1/n}(dx)\Big).$$

In particular, ϕ_n is the characteristic function of a compound Poisson process with intensity n and jump measure $\mu^{1/n}$. We conclude the proof by observing that

$$\begin{aligned}\phi_n(u) &= \exp\Big(n\big((\phi(u))^{1/n} - 1\big)\Big) = \exp\Big(n\big(e^{\frac{1}{n}\log(\phi(u))} - 1\big)\Big)\\ &= \exp\Big(\log(\phi(u)) + no\Big(\frac{1}{n}\Big)\Big) \xrightarrow[n\to\infty]{} \phi(u).\end{aligned}$$

□

Inference for Lévy processes from low frequency observations: a spectral approach

Let $X = (X_t)_{t\geq 0}$ be a jump diffusion process of the form

$$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

with the jump distribution of Y_1 absolutely continuous with respect to the Lebesgue measure.

Suppose that we are interested in the estimation of the Lévy density ρ from n discrete observations of X at the sampling rate $\Delta = 1$:

$$X_1, X_2, \dots, X_n.$$

Let us denote by ϕ_{X_1} the characteristic function of X_1, i.e.

$$\phi_{X_1}(u) = e^{\psi(u)}, \quad \psi(u) = iu\gamma - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1)\rho(x)dx,$$

with $\gamma_0 = \gamma - \int x\mathbf{1}_{|x|\leq 1}\rho(x)dx$ and suppose that the Lévy density ρ satisfies $\int_{\mathbb{R}} x^2\rho(x)dx < \infty$. In particular, from Proposition 1 in Lecture 8 it follows that $\mathbb{E}[X_1^2] < \infty$ and therefore $\mathbb{E}[|X_1|] < \infty$, too. By differentiation (using that $\mathbb{E}[|X_1|] < \infty$) we get

$$\phi'_{X_1}(u) = \exp(\psi(u))\psi'(u)$$

so that, using Lemma 1, we can write

$$\frac{\phi'_{X_1}}{\phi_{X_1}}(u) = \psi'(u).$$

Taking the derivative in u once again (and using the hypothesis $\int_{\mathbb{R}} x^2\rho(x)dx < \infty$) we get

$$\frac{\phi''_{X_1}\phi_{X_1} - (\phi'_{X_1})^2}{\phi^2_{X_1}}(u) = \frac{d}{du}\psi'(u) = -\sigma^2 + i\frac{d}{du}\int_{\mathbb{R}} xe^{iux}\rho(x)dx = -\sigma^2 - \int_{\mathbb{R}} x^2e^{iux}\rho(x)dx.$$

It follows that

$$\int_{\mathbb{R}} x^2e^{iux}\rho(x)dx = \frac{-\phi''_{X_1}(u)\phi_{X_1}(u) + (\phi'_{X_1}(u))^2}{\phi^2_{X_1}(u)} - \sigma^2$$

and therefore, by Fourier inversion,

$$x^2\rho(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\Big(\frac{-\phi''_{X_1}(u)\phi_{X_1}(u) + (\phi'_{X_1}(u))^2}{\phi^2_{X_1}(u)} - \sigma^2\Big)du.$$

If $x \neq 0$,

$$\rho(x) = \frac{1}{2\pi x^2}\int_{\mathbb{R}} e^{-iux}\Big(\frac{-\phi''_{X_1}(u)\phi_{X_1}(u) + (\phi'_{X_1}(u))^2}{\phi^2_{X_1}(u)} - \sigma^2\Big)du. \tag{6}$$

Remark 2. *Under stronger moment conditions on X_1 one may differentiate $\phi_{X_1}(u) = \exp(\psi(u))$ three times thereby eliminating σ^2 from (6).*

Set $Z_j := X_j - X_{j-1}$ and observe that $Z_1, \dots Z_n$ are independent and with the same law as X_1. Let $\widehat{\phi}_n$ be the empirical characteristic function of Z_1, i.e.:

$$\widehat{\phi}_n(t) := \frac{1}{n}\sum_{j=1}^n e^{itZ_j}.$$

By the strong law of large numbers, for any fixed time t, $\widehat{\phi}_n(t)$, $\frac{d}{dt}\widehat{\phi}_n(t)$ and $\frac{d^2}{dt^2}\widehat{\phi}_n(t)$ converge a.s. to $\phi_{X_1}(t)$, $\phi'_{X_1}(t)$ and $\phi''_{X_1}(t)$, respectively. Then a first instinct is to simply use a plug-in device:

$$\frac{1}{2\pi x^2}\int_{\mathbb{R}} e^{-iux}\Big(\frac{-\widehat{\phi}''_n(u)\widehat{\phi}_n(u)+(\widehat{\phi}'_n(u))^2}{\widehat{\phi}^2_n(u)}-\widehat{\sigma}^2\Big)du, \tag{7}$$

where $\widehat{\sigma}^2$ is some estimator of σ^2. However, we cannot use (7) as an estimator of ρ since the integrand may be not integrable and anyway, since $\widehat{\phi}(t)$ appears at the denominator, (7) could be numerically unstable. Several "working" modifications of the natural but "not-working" quantity (7) have been proposed in the literature. As an example, in [5] the suggested estimator is

$$\widehat{\rho}(x)=\frac{1}{2\pi x^2}\int_{\mathbb{R}} e^{-iux}\Big(-\frac{\widehat{\phi}''_n(u)}{\widehat{\phi}_n(u)}\mathbf{1}_{G_u}+\frac{\widehat{\phi}'_n(u))^2}{\widehat{\phi}^2_n(u)}\mathbf{1}_{G_u}-\widehat{\sigma}^2\Big)\phi_\omega(hu)du, \tag{8}$$

where ϕ_ω is the Fourier transform of a kernel function ω and $h>0$ is a bandwidth.

The integral in (8) if finite under the assumption that ϕ_ω has a compact support. The set G_t is defined by

$$G_t=\Big\{|\widehat{\phi}_n(t)|\geq \kappa_n e^{-\frac{\Sigma^2}{2h^2}}\Big\},$$

where $\kappa_n\to 0$ and Σ are suitably chosen.

The result in [5] is then that under appropriate hypothesis on the kernel and an appropriate choice for $\widehat{\sigma}$ (see,e.g. [4]) the rate of convergence of

$$\mathbb{E}[(\widehat{\rho}(x)-\rho(x))^2],\quad \forall x\neq 0$$

is logarithmic. This rate is minimax-optimal for the estimation of the Lévy density ρ at a fixed point $x\neq 0$ over a suitable class of Lévy triplets when the risk is measured by the mean square error.

For an overview about the estimation of the Lévy triplet of a discretely observed Lévy process via the Lévy Khintchine formula see, e.g., [1].

References

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